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A modulus-based nonmonotone line search method for nonlinear complementarity problems

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ABSTRACT

A modulus-based nonmonotone line search method is proposed for nonlinear complementarity problem. The considered problem is first reformulated to a nonsmooth nonlinear system based on the modulus-based decomposition. Then a nonmonotone line search method using simulated annealing rule is generalized to solve the resulting system. The global convergence of the proposed method is established under some suitable assumptions. Preliminary numerical experiments show that, compared with some existing methods, the proposed method is feasible and effective.

1. Introduction

Consider a nonlinear complementarity problem (NCP for short) as follows:

$$\text{find } x \in \mathbb{R}^n, \quad \text{s.t.} \;\; x \geq 0, \;\; f(x) \geq 0, \;\; x^T f(x) = 0, \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear mapping. Evidently, problem (1.1) reduces to a linear complementarity problem (LCP for short) if f is linear. Throughout this paper, the solution set of problem (1.1), denoted by $\mathcal{X}^*$, is assumed to be nonempty.

NCP has received much attentions for various applications arising in mathematical programming, economic equilibrium, engineering design and others [1–3]. Many efficient algorithms have been developed for NCP in literature. Semismooth equations (based on nonlinear complementarity problem function, NCP-function) approaches are state-of-the-art methods. Sun and Qi [4] presented several NCP-functions and investigated their properties, and provided a numerical comparison between the behavior of different NCP-functions. Among the NCP-functions, the Fischer–Burmeister function [5] (FB NCP-function hereafter) is a popular choice since it has some interesting properties. Based on the FB NCP-function, various effective methods were developed. Luca, Facchinei and Kanzow [6] exploited an extended Newton's method and established the global convergence and quadratic convergence rate. Jiang and Qi [7] proposed a hybrid method combining the generalized Newton's method and steepest descent method, and established the global convergence and Q-quadratic convergence rate. Kanzow and Pieper [8] presented a Jacobian smoothing method (which combines nonsmooth Newton method and smoothing methods) to the nonsmooth system resulted by FB NCP-function. Chen, Chen and Kanzow [9] proposed a modified FB NCP-function which has some stronger theoretical properties compared to the classical FB NCP-function. Qi and Yang [10] proposed a Lagrangian globalization algorithm based on NCP-functions. Chen and Pan [11] presented a family of new NCP-functions and a descent method based on these NCP-functions. Chen, Zhang and Fukushima [12] extended NCP-function

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to stochastic LCP. Ma [13] proposed a smoothing and regularization Newton method for NCP with P_0-NCP-function[1], and so on.

Another interesting approach for NCP is a class of methods based on the theory for maximal monotone operators and variational inequalities. In this framework, some effective iteration methods were developed, see [15–18]. He, Liao and Yuan [19] proposed a logarithmic-quadratic proximal method which consists of a relaxed prediction and an explicit correction. Xu, He and Yuan [20] presented a hybrid logarithmic-quadratic proximal method. In fact, the logarithmic-quadratic proximal type methods are essentially as same as the projection and contraction method under the framework for variational inequality.

For LCP, Murty [21] proposed a modulus iterative method by reformulating it to a fixed-point equation. Bai [22] constructed a general framework based on modulus-based matrix splitting (MBMS hereafter), which is very effective in some realistic applications. Zheng and Yin [23] proposed an accelerated MBMS method for large-scale sparse LCP, and established the global convergence when the system matrix is H_+ matrix or positive definite. Moreover, some modified MBMS methods based on the relaxation technique are also effective for LCP, see [24,25].

Recently, some MBMS methods were also developed to solve a class of weak NCP [26–28]. Xie, Xu and Zeng [29] proposed a two-step MBMS method based on fixed-point iteration. The availability of the MBMS type iterative methods mentioned above are dominated by the splitting of the linear components in the NCP under consideration. By this way, the MBMS type iterative methods may loose its advantage if mapping $f(x)$ in NCP has a high nonlinearity. To overcome these drawbacks, in this paper we propose a modulus-based nonmonotone line search method for problem (1.1). Under some suitable assumptions on nonlinear mapping $f(x)$ of problem (1.1), we investigate the global convergence of the proposed method.

The rest of this paper is organized as follows. A modulus-based nonmonotone line search method is proposed in the next section. In section 3, the global convergence of the proposed method is established under some suitable assumptions. In section 4, some preliminary numerical results are presented to show the validity and effectiveness of the proposed method. Section 5 concludes the paper with some final remarks.

2. The modulus-based nonmonotone line search method

Throughout this paper, the Euclidean norm $\|\cdot\|$ is used for vector. The Clarke's subdifferential [30] of f at x is denoted by $\partial f(x)$. The identity matrix is denoted by I, the set $[n] = \{1, 2, \ldots, n\}$ and $\mathbb{Z}_{++}$ is the positive integers set.

Let $u \in \mathbb{R}^n$ and

$$x = |u| + u, \ \ f(x) = |u| - u, \tag{2.1}$$

where absolute value $|\cdot|$ is component-wise. Then we have

$$x \geq 0, \ \ f(x) \geq 0, \quad \text{and} \quad x^T f(x) = 0. \tag{2.2}$$

Consider the non-smooth nonlinear system of equations

$$F(u) = 0, \tag{2.3}$$

where $F(u) := f(u + |u|) + u - |u|$. Then, we can obtain a solution of problem (1.1) via solving the nonlinear equations (2.3). Moreover, we show that the system (2.3) is identical to the problem (1.1) with explicit relation between the solutions of these two systems.

Theorem 2.1. *If $u \in \mathbb{R}^n$ is a solution of Eq. (2.3), then x defined by $x := |u| + u$ is a solution of problem (1.1). Conversely, if $x \in \mathbb{R}^n$ is a solution of the problem (1.1), then u defined by $u := \frac{1}{2}(x - f(x))$ is a solution of Eq. (2.3).*

Proof. Suppose that $u \in \mathbb{R}^n$ is a solution of the nonlinear equations (2.3). Let $x := |u| + u$. By (2.3), we have $f(x) = |u| - u$. Which combined with the fact

$$|u| + u \geq 0, \ \ |u| - u \geq 0, \quad \text{and} \quad (|u| + u)^T (|u| - u) = 0 \tag{2.4}$$

implies that $x = |u| + u$ solves the problem (1.1).

Conversely, suppose that $x \in \mathbb{R}^n$ is a solution of problem (1.1). Let $u := \frac{1}{2}(x - f(x))$. We distinguish the following three cases: (i) when $x_i = 0, \ f_i(x) > 0$ with $i \in [n]$, one have

$$(|u| + u)_i = \frac{f_i(x)}{2} + \frac{-f_i(x)}{2} = 0 = x_i \quad \text{and} \quad (|u| - u)_i = \frac{f_i(x)}{2} - \frac{-f_i(x)}{2} = f_i(x).$$

(ii) when $x_i > 0, \ f_i(x) = 0$ with $i \in [n]$, one have

$$(|u| + u)_i = \frac{x_i}{2} + \frac{x_i}{2} = x_i \quad \text{and} \quad (|u| - u)_i = \frac{x_i}{2} - \frac{x_i}{2} = 0 = f_i(x).$$

(iii) when $x_i = 0, \ f_i(x) = 0$ with $i \in [n]$, one have

$$(|u| + u)_i = 0 = x_i \quad \text{and} \quad (|u| - u)_i = 0 = f_i(x).$$

[1] Mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is a P_0-function if $\max\limits_{1 \leq i \leq n, x_i \neq y_i} (x_i - y_i)(f_i(x) - f_i(y)) \geq 0, \ \forall x, y \in R^n, x \neq y$, see [14].

In summary, we have that $|u|+u=x$ and $|u|-u=f(x)$, which deduces that $f(|u|+u)=|u|-u$. Hence, $u=\frac{1}{2}(x-f(x))$ solves nonlinear system (2.3). □

The problem (1.1) is also equivalent to finding a global minimizer of the unconstrained optimization problem

$$\min_{u\in\mathbb{R}^n} h(u) := \|F(u)\|^2. \tag{2.5}$$

Many researchers investigated the equivalence relation between problem (2.2) and (2.5), and proposed some efficient methods for NCP based on solving its equivalent optimization problem, for example, see Mangasarian and Solodov [31]. However, it is hard to find a global minimizer of problem (2.5) in general since it may be nonconvex. In the proposed method, we only use $h(u)$ as a merit function. It is obvious that, if $h(u)=0$ then $F(u)=0$.

Recently, Dong, Li and Peng [32] proposed a simulated annealing-based BB gradient method (SABB for short) for unconstrained optimization problem, in which the BB step-size [33] and a nonmonotone line search technique based on the improved Wolfe line search [34] are used to get a predictor, and the predictor is accepted as the next iterate via a simulated annealing rule. By generalizing the SABB method to solve nonlinear system (2.3), where $h(u)$ serves as a merit function and $-F(u)$ provides a search direction, we propose a modulus-based nonmonotone line search method for problem (1.1). Algorithm 1 below summarizes the proposed method in details.

Algorithm 1 The modulus-based nonmonotone line search method, MBNLS.

s0. Given a tolerance $\varepsilon>0$, a starting point $u_0\in\mathbb{R}^n$. Let $\gamma\in(0,1)$, $\beta\in(0,1)$, $T_0>0$, $c\in(0,10^{-2})$, $2\le\theta\in\mathbb{Z}_{++}$, $0<\alpha_{\max}\le 10^2$, and $\alpha_0\in(0,\alpha_{\max}]$. Set $k:=0$.
s1. If $h(u_k)\le\varepsilon$, then stop, and accept $x^*=|u_k|+u_k$ as a numerical solution.
s2.] Let $\lambda_k=1$, and

$$z_k = u_k-\lambda_k\alpha_k F(u_k), \tag{2.6}$$

and compute $h(z_k)$. Let $p_k=e^{-\frac{\Delta_k}{T_k}}$, where

$$\Delta_k = h(z_k)-(h(u_k)-c\lambda_k\alpha_k h(u_k)). \tag{2.7}$$

s3. Pick a random number $r_k\in\left[e^{-\theta}, e^{-\frac{1}{\theta}}\right]$. If

$$p_k\ge r_k, \tag{2.8}$$

then $u_{k+1}=z_k$ and go to **s4**. Otherwise, let $\lambda_k:=\beta^{m_k}$ where m_k is the smallest nonnegative integer such that

$$h(u_k-\lambda_k\alpha_k F(u_k))\le h(u_k)-c\lambda_k^2\alpha_k h(u_k), \tag{2.9}$$

and let $u_{k+1}=u_k-\lambda_k\alpha_k F(u_k)$.
s4. Update step-size α_{k+1} by

$$\alpha_{k+1}=\min\left\{\frac{\|s_k\|^2}{s_k^{\mathrm{T}}y_k}, \alpha_{\max}\right\}, \tag{2.10}$$

where $s_k=u_{k+1}-u_k$ and $y_k=F(u_{k+1})-F(u_k)$.
s5. Let $T_{k+1}:=\gamma T_k$, $k:=k+1$, and go to **s1**.

Remark 2.2.

(1) It is easy to derive from (2.8) that

$$h(z_k)\le h(u_k)-c\lambda_k^2\alpha_k h(u_k)-T_k\ln r_k. \tag{2.11}$$

Combining (2.9) and (2.11), we get

$$h(u_{k+1})\le h(u_k)-c\lambda_k^2\alpha_k h(u_k)-\delta_k T_k\ln r_k, \tag{2.12}$$

where $\delta_k=1$ if (2.8) holds, otherwise $\delta_k=0$.

(2) To guarantee the condition (2.12) being well-defined, we set $\alpha_{\max}\le 10^2$ and $c<10^{-2}$ such that $0<c\alpha_k\le c\alpha_{\max}<1$.

3. Convergence analysis

In this section we will prove that, any convergent sub-sequence $\{u_{k_j}\}$ of sequence $\{u_k\}$ generated by Algorithm 1 converges to a solution of system (2.3), which results that sequence $\{x_{k_j}:=|u_{k_j}|+u_{k_j}\}$ converges to a solution of NCP (1.1). Given $u_0\in\mathbb{R}^n$, $\gamma\in(0,1)$, $2\le\theta\in\mathbb{Z}_{++}$ and $T_0>0$, we define the level set

$$\Omega_0 := \left\{u\in\mathbb{R}^n \,\middle|\, h(u)\le h(u_0)+(1-\gamma)^{-1}\theta T_0\right\}$$

and

$$\Omega_0' := \left\{ u' \in \mathbb{R}^n \;\middle|\; u' = |u| + u, u \in \Omega_0 \right\}.$$

If Ω_0 is bounded, Ω_0' is obviously bounded. We need commonly the boundedness of level set associated to some merit function [35]. We make the following assumptions.

Assumption 3.1.

(a) The set Ω_0 is nonempty and bounded;
(b) Function $f(x)$ is locally $\ell-$Lipschitz continuous in an open ball $B_0 := \left\{ u \in \mathbb{R}^n \;\middle|\; \|u\| < r_0 \right\}$ containing $\Omega_0 \cup \Omega_0'$;
(c) The generalized Jacobian operator $\partial F(u)$ is continuous for all $u \in \Omega_0$, and the generalized Jacobian matrix $F'(u) \in \partial F(u)$ is δ-strongly positive definite, i.e., there exists $\delta > 0$ such that $u^T F'(u) u \geq \delta \|u\|^2$, $\forall F'(u) \in \partial F(u), \forall u \in \Omega_0$.

Although merit function $h(u)$ is nonsmooth, the subgradient vector of $h(u)$ is also denoted by $\nabla h(u)$ for convenience.

Remark 3.1. By Assumption 3.1 (c), $F(u)$ is strictly monotone[2] on Ω_0 (Proposition 2.2, [36]). Moreover, $-F(u)^T \nabla h(u) = -F(u)^T F'(u) F(u) \leq 0$ (the equality holds if and only if $F(u) = 0$), which implies that $-F(u)$ is a descent direction of merit function $h(u)$.

Proposition 3.2 (Lemma 2.2, [37])**.** *Suppose that $\mathbb{D} \subset \mathbb{R}^m$ is $\mathcal{C}$-quasiconvex and $F : \mathbb{D} \to \mathbb{R}^n$ is locally ℓ-Lipschitz continuous, then F is $\ell\mathcal{C}$-Lipschitz continuous over set $\mathbb{D}$.*

Lemma 3.3. *If Assumption 3.1 holds, then $F(u)$ and $h(u)$ are Lipschitz continuous in set B_0.*

Proof. let $g(u) = |u| + u$, then $F(u) = f(g(u)) + u - |u|$. Since $f(x)$ is locally ℓ-Lipschitz continuous in set B_0, for all $x \in B_0$ there exists a neighborhood $\mathcal{N}(x, \varepsilon_x) \subset B_0$ such that

$$\|f(x) - f(y)\| \leq \ell \|x - y\|, \quad \forall y \in \mathcal{N}(x, \varepsilon_x).$$

If $u \in B_0$, it follows from $\Omega_0' \subset B_0$ that $g(u) \in B_0$. There exists a neighborhood $\mathcal{N}\big(g(u), \varepsilon_{g(u)}\big) \subset B_0$ such that

$$\|f(g(u)) - f(u')\| \leq \ell \|g(u) - u'\|, \quad \forall u' \in \mathcal{N}\big(g(u), \varepsilon_{g(u)}\big). \tag{3.1}$$

Hence, there is $\varepsilon_u \in \left(0, \frac{\varepsilon_{g(u)}}{2}\right)$ such that for all $u \in B_0$ and $\forall\ v \in \mathcal{N}(u, \varepsilon_u) \subset B_0$, we have $g(v) \in \mathcal{N}\big(g(u), \varepsilon_{g(u)}\big)$. By (3.1) and Triangle-inequality, we get

$$\|f(g(u)) - f(g(v))\| \leq \ell \|g(u) - g(v)\| \leq 2\ell \|u - v\|.$$

Then

$$\begin{aligned} \|F(u) - F(v)\| &= \|[f(g(u)) + u - |u|] - [f(g(v)) + v - |v|]\| \\ &\leq \|f(g(u)) - f(g(v))\| + \|(u - |u|) - (v - |v|)\| \\ &\leq (2\ell + 2)\|u - v\|. \end{aligned} \tag{3.2}$$

Which implies that $F(u)$ is locally $(2\ell + 2)$-Lipschitz continuous in B_0. Note that B_0 is convex, it is easy to deduce that B_0 is 1-quasiconvex[3] By Proposition 3.2, $F(u)$ is $(2\ell + 2)$-Lipschitz continuous in set B_0.

Since B_0 is bounded and $h(u)$ is bounded in B_0, there exists a constant $\ell_{B_0} > 0$ such that $\big|h(u)\big| \leq \ell_{B_0}$, $\forall u \in B_0$. By Cauchy–Schwarz inequality and (3.2), we get

$$\begin{aligned} \big|h(u) - h(v)\big| &= \big| < F(u) - F(v), F(u) + F(v) > \big| \\ &\leq \|F(u) - F(v)\| \|F(u) + F(v)\| \\ &\leq (2\ell + 2)\|u - v\| (\sqrt{h(u)} + \sqrt{h(v)}) \\ &\leq 4(\ell + 1)\sqrt{\ell_{B_0}} \|u - v\|, \ \forall\ u, v \in B_0. \end{aligned} \tag{3.3}$$

Hence $h(u)$ is $\big(4(\ell + 1)\sqrt{\ell_{B_0}}\big)$-Lipschitz continuous in B_0. □

Lemma 3.4. *If Assumption 3.1 holds, then the line search rule (2.9) in Algorithm 1 is well defined.*

Proof. By Assumption 3.1 and Lemma 3.3, we have that $h(u)$ and $F(u)$ are Lipschitz continuous in set B_0. Which follows that $h(u)$ is strictly continuous (Definition 9.1 (b), [38]) in B_0. Then, by the extended mean-value theorem (Theorem 10.48, [38]) and the fact that B_0 is an open convex set, we have

$$h(u_k) - h(u_k - \lambda_k \alpha_k F(u_k)) = 2\lambda_k \alpha_k F(u_k)^T F'(\hat{u}_k) F(\hat{u}_k),$$

---

[2] A mapping F is strictly monotone on $\mathcal{S}$ if for all distinct pairs $x, y \in \mathcal{S}$, $< x - y, F(x) - F(y) >> 0$, see [36].

[3] A set $\mathbb{D} \subset \mathbb{R}^n$ is said to $\mathcal{C}$-quasiconvex with $\mathcal{C} \geq 1$, if every pairs $x, y \in \mathbb{D}$ can be jointed by a curve $\gamma_{\mathbb{D}}$ in $\mathbb{D}$ such that length$(\gamma_{\mathbb{D}}) \leq \mathcal{C}\|x - y\|$, see Section 2.1 in [37].

where $\hat{u}_k = u_k - \tau\lambda_k\alpha_k F(u_k)$ with $\tau \in (0, 1)$, and consequently,

$$\frac{h(u_k) - h(u_k - \lambda_k\alpha_k F(u_k))}{\lambda_k\alpha_k h(u_k)} = \frac{2F(u_k)^T F'(\hat{u}_k)F(\hat{u}_k)}{F(u_k)^T F(u_k)}. \tag{3.4}$$

Taking the limits both sides of (3.4) as $\lambda_k \downarrow 0$, and using Assumption 3.1 (c) and continuity of $F(u)$, we get

$$\lim_{\lambda_k\to 0} \frac{h(u_k) - h(u_k - \lambda_k\alpha_k F(u_k))}{\lambda_k\alpha_k h(u_k)} \geq 2\delta. \tag{3.5}$$

Hence, for all $0 < \epsilon < \delta$, there exists a positive integer N_1 large enough (which deduces that $\lambda_k = \beta^{N_1}$ is small enough) such that

$$\frac{h(u_k) - h(u_k - \lambda_k\alpha_k F(u_k))}{\lambda_k\alpha_k h(u_k)} \geq 2\delta - \epsilon > \delta. \tag{3.6}$$

Since c is a constant, there exists a positive integer N_2 such that $c\lambda_k \leq \delta$ where $\lambda_k = \beta^{N_2}$. Let $m_k = \max\{N_1, N_2\}$, we have that

$$c\lambda_k \leq \delta < \frac{h(u_k) - h(u_k - \lambda_k\alpha_k F(u_k))}{\lambda_k\alpha_k h(u_k)}$$

holds for $\lambda_k = \beta^{m_k}$. Which concludes that line search rule (2.9) in Algorithm 1 is well defined. □

Lemma 3.5. *If sequence $\{u_k\}$ generated by the MBNLS method, then $\{u_k\} \subset \Omega_0$.*

Proof. Let $\eta_k = -\delta_k T_k \ln r_k$ where $\delta_k \in \{0, 1\}$, by $T_k = \gamma^k T_0$ and $\gamma \in (0, 1)$, we have

$$\begin{aligned}
\sum_{k\geq 0} \eta_k &= -\sum_{k\geq 0} \delta_k T_k \ln r_k \leq -\sum_{k\geq 0} T_k \ln r_k \\
&= -\sum_{k\geq 0} \gamma^k T_0 \ln r_k \leq \theta T_0 \sum_{k\geq 0} \gamma^k \\
&\leq (1-\gamma)^{-1}\theta T_0.
\end{aligned} \tag{3.7}$$

The last second inequality follows from $r_k \in \left[e^{-\theta}, e^{-\frac{1}{\theta}}\right]$. By (2.12), we have

$$\begin{aligned}
h(u_{k+1}) &\leq h(u_k) - c\lambda_k h(u_k) - \delta_k T_k \ln r_k \leq h(u_k) - \delta_k T_k \ln r_k \\
&\leq h(u_0) - \sum_{i=0}^{k} T_i \ln r_i \leq h(u_0) + (1-\gamma)^{-1}\theta T_0
\end{aligned}$$

holds for all $k \geq 0$, which implies that $\{u_k\} \subset \Omega_0$. □

Theorem 3.6. *Suppose that Assumption 3.1 holds, and sequence $\{\alpha_k\}$ is generated by the MBNLS method. Then, there exists $\alpha_* > 0$ such that*

$$\alpha_k \geq \alpha_*, \forall k.$$

Proof. By Remark 3.1, $F(u)$ is strictly monotone on Ω_0 which implies

$$\langle u_{k+1} - u_k, F(u_{k+1}) - F(u_k)\rangle > 0.$$

By Lemma 3.3, $F(u)$ is $(2\ell + 2)$-Lipschitz continuous in set B_0. Thus

$$\frac{\|u_{k+1} - u_k\|^2}{\langle u_{k+1} - u_k, F(u_{k+1}) - F(u_k)\rangle} \geq \frac{\|u_{k+1} - u_k\|^2}{\|u_{k+1} - u_k\|\|F(u_{k+1}) - F(u_k)\|} \geq \frac{1}{2\ell + 2},$$

combining with the update scheme (2.10) yields

$$\alpha_k \geq \min\left\{\frac{1}{2\ell + 2}, \alpha_{\max}\right\}, \quad \forall k.$$

Let $\alpha_* = \min\left\{\frac{1}{2\ell+2}, \alpha_{\max}\right\}$, the assertion follows. □

Proposition 3.7. *(Lemma 1, [39]) Let $\{a_k\}$ and $\{b_k\}$ be positive sequences satisfying $a_{k+1} \leq (1 + b_k)a_k + b_k$ and $\sum_{k=0}^{\infty} b_k < \infty$, then $\{a_k\}$ converges.*

The following theorem establishes the global convergence of the MBNLS method.

Theorem 3.8. *Suppose that Assumption 3.1 holds, and sequence $\{u_k\}$ is generated by MBNLS method. Then, there is a subsequence $\{u_{k_j}\} \subset \{u_k\}$ such that*

$$\lim_{k_j\to\infty} \|u_{k_j} - u^*\| = 0,$$

where u^* is a solution of $F(u)=0$. Consequently, $\{x_{k_j}=|u_{k_j}|+u_{k_j}\}$ converges to a solution of nonlinear complementarity problem (1.1).

Proof. By Lemma 3.5, we have $\{u_k\}\subset\Omega_0$. By the continuity of $h(u)$ and boundedness of Ω_0, sequence $\{u_k\}$ has at least a cluster point, we denote it by u^*. There is a convergent sub-sequence $\{u_{k_j}\}\subset\{u_k\}$ such that

$$\lim_{k_j\to\infty}\|u_{k_j}-u^*\|=0. \tag{3.8}$$

By (2.12)

$$h(u_{k+1})\le h(u_k)-c\lambda_k^2\alpha_k h(u_k)+\eta_k,\quad \forall k\ge 0, \tag{3.9}$$

where $\eta_k=-\delta_k T_k\ln r_k>0$. Set $a_k=h(u_k)$ and $b_k=\eta_k$, we have

$$a_{k+1}\le a_k-c\lambda_k^2\alpha_k h(u_k)+b_k\le a_k+b_k\le(1+b_k)a_k+b_k.$$

By Proposition 3.7, sequence $\{h(u_k)\}$ converges and consequently $\{\|F(u_k)\|\}$ converges. Adding (3.9) from $k=0$ to K, we obtain

$$c\sum_{k=0}^{K}\lambda_k^2\alpha_k h(u_k)\le h(u_0)-h(u_{K+1})+\sum_{k=0}^{K}\eta_k\le h(u_0)+\sum_{k=0}^{K}\eta_k. \tag{3.10}$$

Taking limits on both sides of (3.10) as $K\to\infty$, and using Theorem 3.6 and (3.7) we have

$$\sum_{k=0}^{\infty}\lambda_k^2 h(u_k)\le\frac{1}{c\alpha_*}\left[c\sum_{k=0}^{\infty}\lambda_k^2\alpha_k h(u_k)\right]\le\frac{1}{c\alpha_*}\left(h(u_0)+(1-\gamma)^{-1}\theta T_0\right)<\infty.$$

Which deduces that $\lim_{k\to\infty}\lambda_k^2 h(u_k)=0$. Combining the convergence of sequence $\{\|F(u_k)\|\}$, we have

$$\lim_{k\to\infty}\|F(u_k)\|=0\quad\text{or}\quad\liminf_{k\to\infty}\lambda_k=0, \tag{3.11}$$

or both.

If $\lim_{k\to\infty}\|F(u_k)\|=0$, the assertion follows. Otherwise, we have $\liminf_{k\to\infty}\lambda_k=0$. Without loss of generality, we assume that $\lim_{k\to\infty}\lambda_k=0$. Since m_k is the smallest nonnegative integer such that $\lambda_k=\beta^{m_k}\le 1$ satisfies the line search rule (2.9), we have

$$h(u_k-\bar\lambda_k\alpha_k F(u_k))>h(u_k)-c\bar\lambda_k^2\alpha_k h(u_k), \tag{3.12}$$

where $\bar\lambda_k=\frac{\lambda_k}{\beta}$. Thus

$$c\bar\lambda_k>\frac{h(u_k)-h(u_k-\bar\lambda_k\alpha_k F(u_k))}{\bar\lambda_k\alpha_k h(u_k)}. \tag{3.13}$$

Taking limits on the both sides of (3.13) and using (3.5), we get

$$0\ge\lim_{\bar\lambda_k\to 0}\frac{h(u_k)-h(u_k-\bar\lambda_k\alpha_k F(u_k))}{\bar\lambda_k\alpha_k h(u_k)}\ge 2\delta>0, \tag{3.14}$$

which leads to a contradiction. Hence, the case $\liminf_{k\to\infty}\lambda_k=0$ fails, we also obtain $\lim_{k\to\infty}\|F(u_k)\|=0$.

The equality $\lim_{k\to\infty}\|F(u_k)\|=0$ also holds for subsequence $\{u_{k_j}\}$, i.e., $\lim_{k_j\to\infty}\|F(u_{k_j})\|=0$, and equivalently

$$\lim_{k_j\to\infty}F(u_{k_j})=0.$$

By (3.8) and the continuity of $F(u)$, we have $F(u^*)=0$, which implies that subsequence $\{u_{k_j}\}$ converges to a solution of $F(u)=0$. By the equivalence, we have $\{x_{k_j}:=|u_{k_j}|+u_{k_j}\}$ converges to a solution of problem (1.1). □

4. Numerical experiments

In this section, some preliminary numerical results are presented to verify the performance of the proposed MBNLS method. The proposed method is compared with hybrid inexact Logarithmic Quadratic Proximal (LQP for short) [20] and Fisher-Burmeister Semismooth Newton (FBSN for short) [6]. Fifteen test examples are listed in Appendix A. All methods are coded in MATLAB R2016b and run on a personal computer with 1.80 GHz Intel Core i7 and 8 GB RAM.

The main computational cost of each iteration and line search used in these methods are listed in Table 1 for comparisons.

Table 1
The main computational cost of the MBNLS, LQP and FBSN methods.

Methods	Main computational cost	Computational cost in line search
MBNLS	n	n
LQP	$3n$	$3n$
FBSN	$O(n^3)$	$O(n^2)$

Table 2
The stopping criteria for the MBNLS, LQP and FBSN methods.

Methods	iter_max	cput_max	Tolerance
MBNLS	10000	3600s	$\|F(u_k)\| \le 10^{-4}$
LQP			$\left\|F\left(\frac{1}{2}(x_k - f(x_k))\right)\right\| \le 10^{-4}$
FBSN			$\left\|F\left(\frac{1}{2}(x_k - f(x_k))\right)\right\| \le 10^{-4}$

Table 3
The results for FBSN, LQP and MBNLS on medium-scale examples 5.1 $\sim$ 5.4.

Prob	dim	LQP iter/cput/fe/ncpres	FBSN iter/cput/fe/ncpres	MBNLS iter/cput/fe/ncpres
5.1	50^2	18.2/0.0274/39.4/2.1e−04	19/3.6238/39/3.4e−04	25.2/**0.0172**/32.4/4.0e−05
	100^2	19.8/0.1413/42.6/6.3e−04	20.2/60.240/41.4/5.9e−04	25.4/**0.0778**/36/8.2e−04
5.2	50^2	10.8/**0.0152**/24.6/4.0e−06	*/*/*/*	29.4/0.0154/36.8/3.9e−05
	100^2	11/**0.0590**/25/5.7e−06	*/*/*/*	27.8/0.0650/37.4/2.1e−04
5.3	5k	14.2/5.5317/30.4/1.5e−05	3/3.9938/7/2.8e−08	21/**3.0181**/25/3.5e−05
	10k	14.6/22.473/31.2/1.6e−05	3/15.670/7/3.4e−08	22/**12.554**/26/3.6e−05
5.4	5k	*/*/*/*	*/*/*/*	3/**0.3799**/4/1.6e−08
	10k	*/*/*/*	*/*/*/*	2/**1.0247**/3/1.2e−02

We define the residual of nonlinear complementarity problem (ncpres for short) as follows

$$\textbf{ncpres} := \max\left\{\|\min(x_k, 0)\|,\ \ \|\min(f(x_k), 0)\|,\ \ |x_k^T f(x_k)|\right\}. \tag{4.1}$$

The stopping criteria of the MBNLS, LQP and FBSN are set in Table 2. The corresponding algorithm stops whenever one of these criterias meets.

Remark 4.1. By Theorem 2.1, x solves NCP (1.1) if and only if $u = \frac{1}{2}(x - f(x))$ solves the nonsmooth nonlinear equations $F(u) = 0$. Hence, $\|F(u_k)\|$ and $\left\|F\left(\frac{1}{2}(x_k - f(x_k))\right)\right\|$ can be adapted as the measurement in stopping criterion.

In the implementation of all methods, the initial point is set to $x_0 = rand(n, 1)$. The algorithmic-parameters are set as follows:

(1) For FBSN, set $\beta = 0.2, \rho = 0.01, p = 2.2$ as suggested in [6];
(2) For LQP, set $\beta_0 = 1, \eta = 0.95, \mu = 0.01, \gamma = 1.9$ and $\sigma = 1$ as suggested in [20];
(3) For MBNLS, we set $\alpha_0 = 1, \alpha_{\max} = 10^2, c = 10^{-4}, \beta = 0.618, \theta = 20, T_0 = 10^3, \gamma = 0.9$.

For easy reference, the notations used in the numerical results, i.e., Tables 3 and 4, are interpreted as follows:
Prob: test example index; **dim**: dimension of test example, 1k $:= 1.0 \times 10^3$;
iter: total number of iterations; **cput**: CPU time in seconds;
fe: total number of function evaluations; **ncpres**: residual error defined by (4.1) ;
/*/*/: **iter** > 10000 or **cput** $> 3600s$ or **ncpres**$> 10^{-1}$;
The proposed MBNLS method is compared with FBSN and LQP on the medium-scale test Examples 5.1–5.4, and compared with LQP on large-scale test Examples 5.5–5.15. All results are the average over five runs. For fairness, the method who has the least cpu time (cput in seconds) is selected as the winner and highlighted by red letters.

From Tables 3 and 4 one can find that, the MBNLS method is selected as the winner in most cases. In the sense, we conclude that the performance of MBNLS method is superior to the FBSN and LQP method.

Table 4
The results for LQP and MBNLS on large-scale examples 5.5 ∼ 5.15.

Prob	dim	LQP iter/cput/fe/ncpres	MBNLS iter/cput/fe/ncpres
5.5	5k	12/0.0046/25/1.3e−04	40/**0.0030**/41/1.4e−04
	50k	14/0.0372/29/1.1e−04	81/**0.0297**/82/3.1e−04
	500k	15/**0.4938**/31/3.2e−04	170/1.8591/171/6.6e−04
5.6	5k	12/0.0053/26/2.9e−06	12/**0.0012**/13/4.5e−06
	50k	13/0.0412/28/1.2e−05	14/**0.0063**/15/1.4e−05
	500k	14/0.8378/30/1.4e−05	20/**0.4002**/21/2.1e−05
5.7	5k	2/0.0010/6/0.0e+00	4/**0.0002**/5/0.0e + 00
	50k	2/0.0074/6/0.0e+00	4/**0.0017**/5/0.0e + 00
	500k	2/0.0860/6/0.0e+00	3/**0.0457**/5/0.0e + 00
5.8	5k	1/0.0007/4/0.0e+00	3/**0.0004**/5/0.0e + 00
	50k	1/0.0056/4/0.0e+00	3/**0.0036**/5/0.0e + 00
	500k	1/0.0677/4/0.0e+00	3/**0.0606**/5/0.0e + 00
5.9	5k	16/0.0069/34/5.2e−09	5/**0.0004**/5/0.0e + 00
	50k	18/0.0575/38/3.1e−09	3/**0.0023**/5/0.0e + 00
	500k	19/1.0794/40/7.9e−09	3/**0.0705**/5/0.0e + 00
5.10	5k	12/0.0042/25/1.3e−04	40/**0.0028**/41/1.4e−04
	50k	14/0.0331/29/1.1e−04	81/**0.0269**/82/3.1e−04
	500k	15/**0.5061**/31/3.2e−04	170/1.8860/171/6.6e−04
5.11	5k	666.8/0.3102/1429.6/4.6e−08	76.6/**0.0078**/79.8/2.7e−10
	50k	927/3.4008/2051/4.5e−08	89.4/**0.0539**/92.4/3.6e−11
	500k	1217.8/88.162/2727.6/4.0e−08	16/**0.3883**/19/2.4e−09
5.12	5k	713.6/0.2685/1513.4/8.1e−08	156.2/**0.0115**/159.2/2.8e−09
	50k	962.2/2.7590/2106.8/8.1e−08	116.2/**0.0484**/129.2/1.9e−08
	500k	1184.6/52.993/2680/9.0e−08	45/**0.6651**/52.8/1.4e−08
5.13	5k	14.6/0.0191/34.2/2.3e−06	19.4/**0.0074**/26.4/5.6e−05
	50k	16/0.1677/37.8/2.4e−05	21/**0.0787**/28/3.9e−05
	500k	20.2/2.0118/47.4/1.7e−05	20/**0.7983**/27/3.9e−05
5.14	5k	48.6/0.0182/99.2/0.0e+00	3/**0.0004**/6/0.0e + 00
	50k	91.2/0.2641/184.4/0.0e+00	3/**0.0024**/6/0.0e + 00
	500k	165.6/10.649/333.2/0.0e+00	3/**0.1012**/6/0.0e + 00
5.15	5k	11.2/2.2956/24.4/2.5e−02	7.8/**0.5415**/8.8/2.1e−04
	50k	11.6/143.00/25.2/3.8e−02	8/**31.884**/9/2.1e−03
	300k	*/*/*/*	8/**1991.1**/9/1.3e−02

5. Conclusions

In this paper, we proposed a modulus-based nonmonotone line search method for nonlinear complementarity problem. By a modulus-based decomposition, the nonlinear complementarity problem (1.1) is reformulated to a nonlinear nonsmooth system. We extended a nonmonotone line search method to solve the resulting system, and consequently, it solves the nonlinear complementarity problem. The global convergence of the proposed method is established under some suitable assumptions. Numerical results show that, compared with the FBSN and LQP, the proposed MBNLS method outperforms in most cases.

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Appendix A.

Example 5.1 [29]**.** Let $f(x) = Ax + \Phi(x) + q$, where $A = \text{diag}(-I, H, -I) \in R^{n\times n}$, with $H = \text{diag}(-1, 4, -1) \in R^{\sqrt{n}\times\sqrt{n}}$, $q = (-1, 1, -1, 1, \ldots)^T$, and $\Phi(x) = \text{vec}\{\Phi_i(x_i)\}$ is a diagonal mapping with endowed component $\Phi_i(x_i) = x_i/(1 + x_i)$, $i = 1, 2, \ldots, n$.

Example 5.2 [29]**.** Let $f(x) = Au + \Phi(x) + q$, where $A = \text{diag}(-1.5I, H, -0.5I) \in R^{n\times n}$, with $H = \text{diag}(-1.5, 4, -0.5) \in R^{\sqrt{n}\times\sqrt{n}}$, $q = (1, -1, 1, -1, \ldots)^T$, and $\Phi(x) = \text{vec}\{\Phi_i(x_i)\}$ is a diagonal mapping with endowed component $\Phi_i(x_i) = \arctan(x_i)$, $i = 1, 2, \ldots, n$.

Example 5.3 [40]**.** The function *f*(*x*) is given by $f(x) = Ax + g(x)$, where $A = \mathrm{diag}(-1, 2, -1)$, $g(x) = \mathrm{vec}\{e^{x_i} - 1\}$, $i = 1, 2, \ldots, n$.

Example 5.4 [39]**.** The function *f*(*x*) is endowed with the component as follows:

$$f_1(x) = x_1 - e^{\cos\left(\frac{x_1+x_2}{n+1}\right)}, \quad f_n(x) = x_n - e^{\cos\left(\frac{x_{n-1}+x_n}{n+1}\right)},$$
$$f_i(x) = x_i - e^{\cos\left(\frac{x_{i-1}+x_i+x_{i+1}}{n+1}\right)}, \quad i = 2, 3, \ldots, n-1.$$

Example 5.5 [40]**.** The function *f*(*x*) is endowed with the component as follows:

$$f_i(x) = x_i - \sin(x_i), \quad i = 1, 2, \ldots, n.$$

Example 5.6 [39]**.** The function *f*(*x*) is endowed with the component as follows:

$$f_i(x) = \min\left(\min\left(|x_i|, x_i^2\right), \max\left(|x_i|, x_i^3\right)\right), \quad i = 1, 2, \ldots, n.$$

Example 5.7 [39]**.** The function *f*(*x*) is endowed with the component as follows:

$$f_i(x) = e^{x_i} - 1, \quad i = 1, 2, \ldots, n.$$

Example 5.8 [39]**.** The function *f*(*x*) is endowed with the component as follows:

$$f_i(x) = x_i - \frac{1}{n}x_i^2 + \frac{1}{n}\sum_{k=1}^{n} x_k + i, \quad i = 1, 2, \ldots, n.$$

Example 5.9 [41]**.** The function *f*(*x*) is endowed with the component as follows:

$$f_1(x) = e^{x_1} - 1,$$
$$f_i(x) = e^{x_i} + x_{i-1} - 1, \quad i = 2, 3, \ldots, n.$$

Example 5.10 [42]**.** The function *f*(*x*) is endowed with the component as follows:

$$f_i(x) = x_i - \sin(|x_i|), \quad i = 1, 2, \ldots, n.$$

Example 5.11 [43]**.** The function *f*(*x*) is endowed with the component as follows:

$$f_1(x) = e^{x_1} - 1,$$
$$f_i(x) = \tfrac{i}{10}(e^{x_i} + x_{i-1} - 1), \quad i = 2, 3, \ldots, n.$$

Example 5.12 [43]**.** The function *f*(*x*) is endowed with the component as follows:

$$f_i(x) = \frac{i}{10}(e^{x_i} - 1), \quad i = 1, 2, \ldots, n.$$

Example 5.13 [43]**.** The function *f*(*x*) is endowed with the component as follows:

$$\begin{aligned} f_1(x) &= 3x_1^3 + 2x_2 - 5 + \sin(x_1 - x_2)\sin(x_1 + x_2), \\ f_i(x) &= \quad -x_{i-1}e^{x_{i-1}-x_i} + x_i(4 + 3x_i^2) + 2x_{i+1} \\ &\qquad + \sin(x_i - x_{i+1})\sin(x_i + x_{i+1}) - 8, \quad i = 2, 3, \ldots, n-1, \\ f_n(x) &= -x_{n-1}e^{x_{n-1}-x_n} + 4x_n - 3. \end{aligned}$$

Example 5.14 [43]**.** The function *f*(*x*) is endowed with the component as follows:

$$f_1(x) = (3 - 0.5x_1)x_1 - 2x_2 + 1, \quad f_n(x) = (3 - 0.5x_n)x_n - x_{n-1} + 1,$$
$$f_i(x) = (3 - 0.5x_i)x_i - x_{i-1} - 2x_{i+1} + 1, \quad i = 2, 3, \ldots, n-1.$$

Example 5.15 [43]**.** The function *f*(*x*) is endowed with the component as follows:

$$f_i(x) = x_i - \left(1 - \frac{c}{2n}\sum_{j=1}^{n}\frac{\mu_i x_j}{\mu_i + \mu_j}\right)^{-1},$$

with $c \in [0, 1)$ and $\mu_i = n^{-1}(i - 0.5)$, for $i = 1, 2, \ldots, n$. (we set $c = 0.9$ as suggested in [43]).

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